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LETTER TO THE EDITOR

The exact invariant density for a cusp-shaped return map

P C Hemmer

Institut for teoretisk fysikk, NTH, Universitetet i Trondheim, N-7034 Trondheim, Norway

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Abstract. The square-root iteration $x_{t+1} = 1 - 2|x_t|^{1/2}$ has a linear invariant density and a characteristic Liapunov exponent $\bar{\lambda} = \frac{1}{2}$.

By plotting successive values of the maxima for a selected variable in the Lorenz model against the previous maximum, a cusp-shaped return map is obtained (see figure 15, and references, in Ott (1981)). The map has, at least approximately, an algebraic form

$$x_{t+1} = 1 - \mu |x_t|^z \tag{1}$$

near the cusp, with $0 < z < 1$.

Figure 1 shows the perhaps simplest example of a return map of this qualitative nature, namely

$$x_{t+1} = 1 - 2(|x_t|)^{1/2}. \tag{2}$$

By the choice $\mu = 2$ the interval $[-1, 1]$ is mapped onto itself rather than onto a proper sub-interval.

The non-invertible map (2) has a slope nowhere less than unity in absolute value. Hence it is ergodic with a unique invariant density $\rho(x)$ (Lasota and Yorke 1973). In the present note we draw attention to the fact that, for (2), $\rho(x)$ and the characteristic Liapunov exponent, $\bar{\lambda}$, can be obtained explicitly.

The invariant density $\rho(x)$ satisfies the Frobenius–Perron equation, in this case

$$\rho(x) = \int_{-1}^{+1} \mathrm{d}y\, \delta(x - 1 + 2|y|^{1/2}) \rho(y). \tag{3}$$

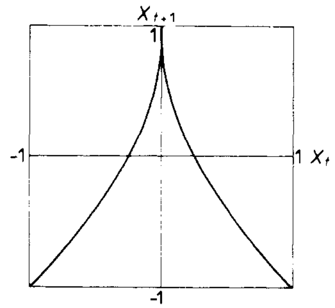

Figure 1. The square-root map (2).

Then

$$\rho(1-2x)=\tfrac{1}{2}\int_{-1}^{+1} \mathrm{d}y\,\delta(|y|^{1/2}-x)\rho(y)=x\rho(x^2)+x\rho(-x^2). \tag{4}$$

One sees by inspection that

$$\rho(x)=\tfrac{1}{2}(1-x) \tag{5}$$

is a solution of (4), and since the solution is unique, it is *the* invariant density. The factor $\frac{1}{2}$ in (5) normalises the integrated density to unity.

To illustrate the deviations of a finite-time value distribution from the invariant (limiting) density $\rho(x)$ we show, in figure 2, a histogram of the value distribution for 2^{18} iterations.

The Liapunov characteristic exponent $\bar{\lambda}$, which is a measure of the exponential divergence of two nearby itineraries, can for the map $x_{t+1}=f(x_t)$ be defined through a time series,

$$\bar{\lambda}=\lim_{T\to\infty} T^{-1}\sum_{t=1}^{T} \ln|f'(x_t)|, \tag{6}$$

and was calculated numerically (as a finite time series) by Mo and Hemmer (1983) for the iteration (1) in the chaotic regime as function of μ and z. The numerical value

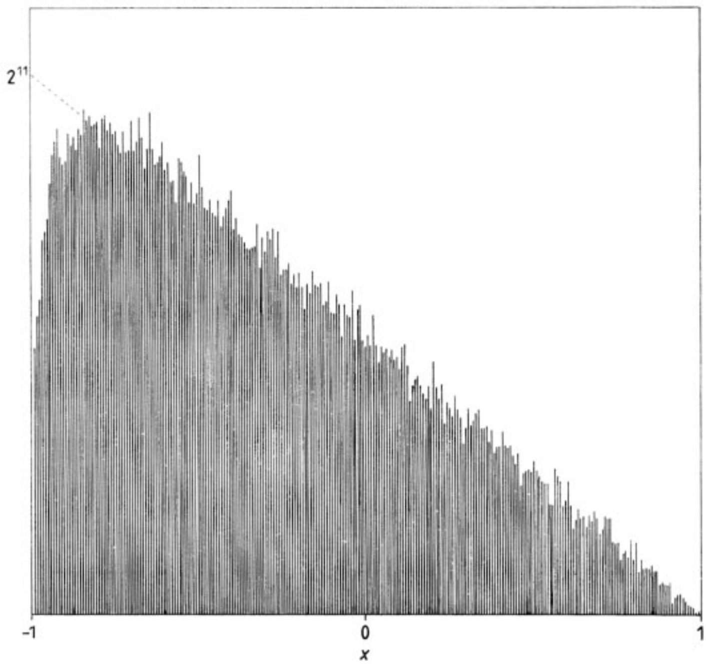

Figure 2. Value distribution of 2^{18} iterates of the square-root map (2), initiated by $x_0=\frac{1}{2}$. The histogram is based on a division of the interval $(-1,+1)$ into 2^8 equal compartments. The linearly decreasing limiting distribution is indicated by a broken line. The slow convergence to the limiting distribution for x close to the marginally stable fixed point at -1 is not surprising, since visits to this region are very few (but extended). (I am indebted to Stig Hemmer for providing the data for this figure).

(accurate to the digits shown) for the special case, $\mu = 2$, $z = \frac{1}{2}$ was $\bar{\lambda} = 0.50$. When one possesses the invariant density, one can alternatively calculate $\bar{\lambda}$ as an ensemble average,

$$\bar{\lambda} = \int \mathrm{d}x\, \rho(x) \ln|f'(x)|. \tag{7}$$

For the present map one obtains

$$\bar{\lambda} = \int_{-1}^{+1} \tfrac{1}{2}(1-x) \ln|x|^{-1/2}\, \mathrm{d}x = \tfrac{1}{2}, \tag{8}$$

confirming the numerical result.

The variation of the Liapunov exponent $\bar{\lambda}$ with μ and z for the more general cusp-shaped map (1) is described by Mo and Hemmer (1983).

The correlation function

$$C(t) = \langle (x_t - \langle x_t \rangle)(x_0 - \langle x_0 \rangle) \rangle / \langle (x_0 - \langle x_0 \rangle)^2 \rangle,$$

averages (over x_0) taken with the measure (5), decay relatively slowly for the present map, starting with $C(0) = 1$, $C(1) = 4/7$, $C(2) = 38/77$, $C(3) = 328\,633/749\,059$.

References

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